

Topology in lattice QCD

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Abstract: The status of topology on the lattice is reviewed. Recent results show that the topological susceptibility χ can be unambiguously determined. Different methods, if properly implemented, give results consistent with each other. For $SU(3)$ the Witten - Veneziano prediction is confirmed. Preliminary results for full QCD are presented. The problem there is that the usual hybrid montecarlo algorithm has severe difficulty to thermalize topology. Possible ways out are under study.

1 Introduction

Topology plays a fundamental role in QCD. The key equation is the anomaly of the $U_A(1)$ axial current[1]

$$\partial^\mu J_\mu^5(x) = 2N_f Q(x) \quad (1)$$

$J_\mu^5 = \sum_{f=1}^{N_f} \bar{\psi}_f \gamma^5 \gamma_\mu \psi_f$ is the singlet axial current. N_f is the number of light flavours, and

$$Q(x) = \frac{g^2}{32\pi^2} \sum_{a,\mu,\nu} G_{\mu\nu}^a \tilde{G}^{a\mu\nu} \quad (2)$$

the topological charge density. $\tilde{G}_{\mu\nu}^a = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma a}$ is the dual tensor field strength. At the classical level $\partial^\mu J_\mu^5 = 0$: the right hand side of eq.(1) comes from quantum effects, whence the name of anomaly. $Q(x)$ is related to the Chern current $K_\mu(x) = \frac{g^2}{16\pi^2} \epsilon^{\mu\alpha\beta\gamma} A_\alpha^a \left(\partial_\beta A_\gamma^a - \frac{g}{3} f^{abc} A_\beta^b A_\gamma^c \right)$ as follows

$$\partial^\mu K_\mu = Q(x) \quad (3)$$

Eq.(1) can then be written

$$\partial^\mu (J_\mu^5 - 2N_f K_\mu) = 0 \quad (4)$$

from which Ward identities are derived[2].

A number of physical consequences follow from eq.'s (1), (4).

1) The $U_A(1)$ problem. This problem goes back to Gellmann's free quark model, from which the symmetries of hadron physics were abstracted. In that model $U_A(1)$ is a symmetry, i.e. $\partial^\mu J_\mu^5 = 0$. This implies either the existence of parity doublets in the hadron spectrum, if the symmetry is Wigner, or, if it is Goldstone, Weinberg's inequality[3] for the η' mass $m_{\eta'} \leq \sqrt{3}m_\pi$. Neither is true in nature.

In QCD $U_A(1)$ is not a symmetry, due to the anomaly (1) and in principle there is no $U_A(1)$ problem. The anomaly can however explain the high value of the η' mass, as suggested by a $1/N_c$ expansion. The idea is that in the limit $N_c \rightarrow \infty$, with $g^2 N_c$ fixed the theory preserves all the main physical features of

QCD, like confinement[4]. Non leading terms act as a perturbation on this limiting model. As $N_c \rightarrow \infty$ the anomaly disappears, being $\mathcal{O}(g^2)$ [eq.(2)], and the η' is the Goldstone boson of the spontaneously broken $U_A(1)$. The effect of the anomaly is to shift the Goldstone pole from zero mass. This shift can be computed by use of the Ward identities (4) giving[2]

$$\frac{2N_f}{f_\pi^2} \chi = m_\eta^2 + m_{\eta'}^2 - 2m_K^2 \quad (5)$$

$\chi = \chi(q^2)|_{q=0}$ is the topological susceptibility of the vacuum at $N_c = \infty$, or

$$\chi(q^2) = \int d^4x e^{iqx} \langle 0 | T(Q(x)Q(0)) | 0 \rangle_{quenched} \quad (6)$$

$N_c = \infty$ implies quenched approximation, fermion loops being $\mathcal{O}(g^2 N_f)$.

Eq.(5) predicts, within $\mathcal{O}(1/N_c)$ accuracy

$$\chi = (180 \text{ MeV})^4 \quad (7)$$

To be definite a prescription must be given for the singularity in the product $Q(x)Q(0)$ in eq.(6) as $x \rightarrow 0$. The prescription which brings to eq.(5) is[2, 5]

$$\chi = \int d^4(x-y) \partial_\mu^x \partial_\nu^y \langle 0 | T(K_\mu(x)K_\nu(y)) | 0 \rangle \quad (8)$$

By the prescription (8) δ like singularities in the product $K_\mu(x)K_\nu(y)$ as $x \rightarrow y$ are eliminated after integration. In any regularization scheme only the multiplicative renormalization of K_μ is thus left. Eq.(5) can be verified on the lattice, where $\chi_{quenched}$ can be computed. This computation provides a check of the $1/N_c$ expansion, which is a fundamental issue.

2) The behaviour of χ across the deconfining phase transition is relevant to understanding the structure of QCD vacuum[6].

3) The $U_A(1)$ Ward identities predict that in full QCD χ behaves in the chiral limit linearly in the quark mass

$$\chi \simeq \frac{1}{N_f} \langle \sum_f m_f \bar{\psi}_f \psi_f \rangle \quad (9)$$

4) Another important quantity to determine on the lattice is $\chi' = \frac{d}{dq^2} \chi(q^2) \Big|_{q^2=0}$. This determination in full QCD is relevant to understand the spin content of the proton[7].

5) The spin content of the proton can be determined by use of Eq.(1), as will be discussed in detail in the following.

A lattice regulator $Q_L(x)$ of the operator $Q(x)$, eq.(2) is needed. The regularized matrix elements are then determined by numerical simulations, and out of them the continuum physical quantities can be extracted by proper renormalization.

2 Defining $Q(x)$ on the lattice.

According to the general rule any gauge invariant operator on the lattice $Q_L(x)$, such that in the formal limit $a \rightarrow 0$

$$Q_L(x) = Q(x)a^4 + \mathcal{O}(a^6) \quad (10)$$

is a possible regulator of topological charge density.

A large arbitrariness by higher order terms exists in the choice of Q_L , which can be used to improve the operator in the sense which will be discussed below. A prototype choice for Q_L is

$$Q_L^{(0)}(x) = -\frac{1}{32\pi^2} \sum_{\pm\mu\nu\rho\sigma} Tr \left[\Pi_{(0)}^{\mu\nu}(x) \Pi_{(0)}^{\rho\sigma}(x) \right] \quad (11)$$

where $\Pi_{(0)}^{\mu\nu}$ is the usual plaquette operator in the plane μ, ν . $Q_L^{(0)}$ obeys eq.(10), and differs by $\mathcal{O}(a^6)$ from any other choice. We will make use of a recursive improving of the operator (11)[8]. We shall define $Q_L^{(i)}(x)$ ($i = 1, 2, \dots$) by a formula similar to (11), with $\Pi_{(0)}^{\mu\nu}\Pi_{(0)}^{\rho\sigma}$ replaced by $\Pi_{(i)}^{\mu\nu}\Pi_{(i)}^{\rho\sigma}$. For $i = 1$ each link is replaced by a smoothed link, for $i = 2$ each link of $\Pi_{(1)}$ is smoothed again and so on[8]. An alternative definition is the so called geometrical charge[9, 10], which again obeys the constraint eq.(10).

Any Q_L , in the limit $a \rightarrow 0$, will be a mixing of all the continuum operators with the same quantum numbers and with lower or equal dimension. In quenched QCD the only pseudoscalar with dimension ≤ 4 is $Q(x)$ itself and therefore[11]

$$Q_L(x) \underset{a \rightarrow 0}{\simeq} Z Q(x) \quad (12)$$

Z can be determined by a non perturbative procedure known as “heating”[12]. The idea is to measure Q_L on a state with a definite value of Q , which then determines Z . To do that an instanton is put by hand on the lattice, and quantum fluctuations at a given value of $\beta = 2N/g^2$ are produced numerically by the usual updating algorithm. A plateau after a few heating steps signals thermalization of these fluctuations, while the topological content of the configuration takes a much longer time to be changed, as can be directly tested: the values of Z for different operators Q_L at different β ’s are shown in Table 1.

	Op	$\chi_L \cdot 10^{-5}$	$M \cdot 10^{-5}$	M/χ_L	Z	$\chi^{1/4}$ MeV
$N_c = 3$ $\beta = 5.9$ $a^{-1} = 1.74(4)$ GeV	Q_0	2.72(6)	2.50(15)	.9	.12(4)	167(36)
	Q_1	2.48(5)	.88(6)	.35	.36(2)	178(6)
	Q_2	3.51(7)	.71(6)	.2	.48(2)	175(6)
$N_c = 3$ $\beta = 6.1$ $a^{-1} = 2.17(5)$ GeV	Q_0	2.14(4)	1.97(10)	.9	.18(4)	196(40)
	Q_1	1.12(2)	.47(3)	.42	.41(2)	178(6)
	Q_2	1.39(2)	.33(2)	.24	.54(2)	175(6)
$N_c = 2$ $\beta = 2.44$ $a^{-1} = 1.92$ GeV	Q_0	3.38(8)	2.93(7)	.87	.18(6)	208(23)
	Q_1	3.26(8)	1.45(4)	.44	.405(13)	197(8)
	Q_2	4.92(14)	1.34(5)	.27	.568(16)	197(8)
	Q_{Geo}	69.7 ± 1.1	52(2)	.75	.918(57)	231(13)
$N_c = 2$ $\beta = 2.57$ $a^{-1} = 2.91(9)$ GeV	Q_0	2.32(5)	2.20(3)	.95	.240(26)	197(27)
	Q_1	1.01(5)	.44(2)	.45	.507(9)	200(8)
	Q_2	1.16(6)	.117(5)	.16	.675(8)	198(7)
	Q_{Geo}	16.6(3)	13.26(23)	.80	.937(26)	228(12)

Table 1. χ_L, M, χ, Z, a for quenched $SU(3)$ and $SU(2)$ at various β .

3 Measuring χ

The lattice susceptibility is defined as

$$\chi_L = \sum_x \langle Q_L(x) Q_L(0) \rangle = \frac{\langle Q_L^2 \rangle}{V} \quad (13)$$

χ_L is a positive quantity by definition. However in the euclidean region, for $x \neq 0$, $\langle Q(x) Q(0) \rangle$ is a negative quantity, by reflection positivity, Q being odd under time reflection. This can be checked on the lattice and the result is shown in fig.1. If the operator is smeared over a region of size s , one expects $\langle Q(x) Q(0) \rangle$ to be negative at distances $|x| \geq s$. The peak at $x = 0$ is essential to make χ positive, and hence the prescription for the product at $x = 0$ is essential. In general, as a consequence of operator product expansion around $x = 0$, χ_L will mix with the continuum operators with the same quantum numbers and lower or equal dimension[13]

$$\chi_L = Z^2 \chi a^4 + M(\beta) + \mathcal{O}(a^6) \quad (14)$$

with

$$M(\beta) = \overline{Z}(\beta) a^4 \langle \frac{\beta(g)}{g} G_{\mu\nu}^a G_{\mu\nu}^a \rangle + \overline{\overline{Z}}(\beta) a^4 \langle m \bar{\psi}(x) \psi(x) \rangle + P(\beta) \langle I \rangle \quad (15)$$

In the quenched approximation the second term in eq.(15) will be absent.

The first term in eq.(14) corresponds to the prescription (8) and renormalizes multiplicatively. Since $\int d^4x \partial_\mu K^\mu = Q$ the prescription (8) implies that χ must be zero on the sector $Q = 0$. $M(\beta)$ can be obtained by measuring χ_L on that sector. This is done by the same heating technique[12] used to determine Z : a sample of configurations belonging to the $Q = 0$ sector is produced by heating the flat (zero field) configuration, in such a way that the topological charge is not changed. From eq.(14) then

$$\chi = \frac{\chi_L - M(\beta)}{a^4(\beta) Z^2(\beta)} \quad (16)$$

All the quantities in the r.h.s. of eq.(16) depend on the choice of the operator, and/or on the choice of the action, but the result must be independent of them. This appears from Table 1, and from fig.2,3.

A good operator is such that $|M(\beta)|/|\chi_L| \ll 1$ so that most of the observed signal is physical. $Z \simeq 1$ is also desirable. The quality of different Q_L 's can be appreciated from Table 1.

The behaviour of χ at deconfinement is shown in fig.2 [14], where $SU(2)$ and $SU(3)$ can be directly compared.

The existence on lattice of a nontrivial Z was first realized in ref.13. Eq.(16) was first introduced in ref.11, where, however, the renormalization constants were determined by perturbation theory.

The heating technique[12] finally allowed a non perturbative determination of them.

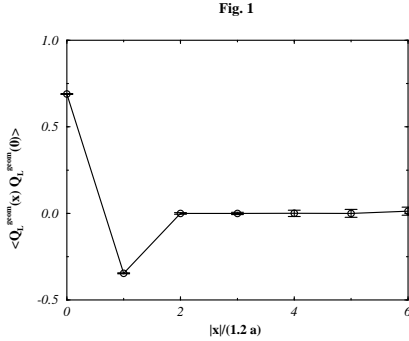


fig.1 Correlation $\langle Q(x)Q(0) \rangle$ for the geometric topological charge density.

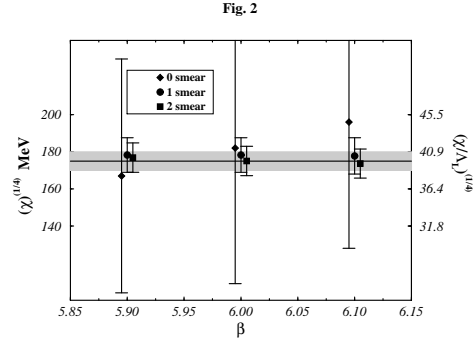


fig.2 χ for quenched $SU(3)$. Diamonds, circles and squares correspond to 0,1 and 2 - smeared operators.

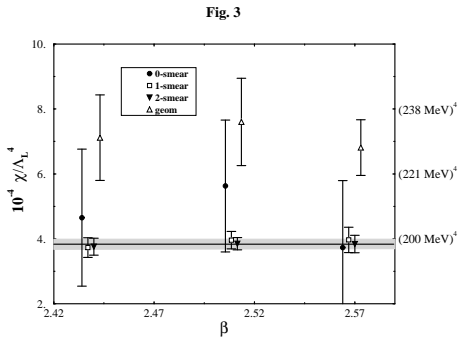


fig.3 χ for quenched $SU(2)$.

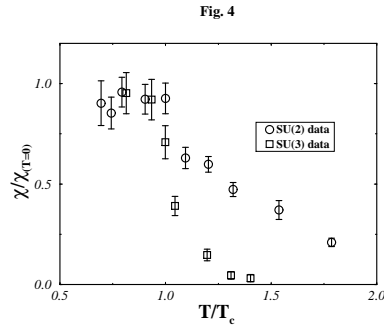


fig.4 χ across the deconfinement transition for $SU(2)$ (circles) and $SU(3)$ (squares).

4 Full QCD¹⁷.

We have used the same technique to determine χ in full QCD with staggered fermions at $m_Q = 0.01$ and $m_Q = 0.02$ at $\beta = 5.35$. The lattice was $16^3 \times 24$. At this β value $a = 0.115(2)$ fm.

We do not have enough precision to test the dependence on m_Q [eq.(9)]. At $am = 0.01$ we get $\chi = (110 \pm 6 \pm 2)^4 \text{ MeV}^4$. The first error is obtained by jake-knife technique. The second comes from the error in the determination of the scale. The expected value is $\chi \simeq (109 \text{ MeV})^4$.

As we will soon discuss these errors are highly underestimated.

A similar analysis can be done for χ' . On general grounds

$$\chi' = Z^2 \chi' a^2 + M'(\beta) + \mathcal{O}(a^4) \quad M'(\beta) = P'(\beta) \langle I \rangle \quad (17)$$

χ' only mixes to the identity operator, since no other gauge invariant operator of dimension ≤ 2 exists.

The technique used to determine χ' is to improve the operator enough, so that $M'(\beta)$ will be negligible as compared to the first term. Z is known from the analysis of χ .

A preliminary result is

$$\chi' = (258 \pm 100) \text{ MeV}^2 \quad \sqrt{\chi'} = (16 \pm 3) \text{ MeV} \quad (18)$$

to be compared with the value computed from SVZ rules[7] $\sqrt{\chi'} = (22.3 \pm 4.8) \text{ MeV}$. Here the error coming from the normalization is much smaller than the statistical error.

Again here the error is underestimated. The reason can be in the history of the topological charge along the updating process is shown. The hybrid montecarlo algorithm is very slow in thermalizing topology[16]. The sampling is very bad and corresponds in fact to a much smaller number of independent configurations.

The same inconvenience affects the determination of the spin content of the proton which will be discussed in the next section. The real error is then larger than the one estimated on the present ensemble of configurations.

5 The spin content of the proton.

The matrix element of the singlet axial current between proton states can be written as

$$\langle \vec{p}' s' | J_\mu^5(0) | \vec{p} s \rangle = \bar{u}(\vec{p}' s') \{ G_1(k^2) \gamma^5 \gamma_\mu + G_2(k^2) \gamma^5 k_\mu \} u(\vec{p} s) \quad (19)$$

$k = p - p'$ is the momentum transfer. $G_1(0) \equiv \Delta\Sigma$ is related to the integral of the spin dependent structure function $g_1(x, q^2)$ of deep inelastic scattering of leptons off protons. In naive parton picture $\Delta\Sigma = \Delta u + \Delta d + \Delta s$ is the fraction of the proton spin carried by the quarks and is expected to be $\Delta\Sigma = 0.7$. Experiments[18] show that it is much smaller, $\Delta\Sigma = 0.21 \pm 0.10$, and this fact is usually referred as spin crisis. By use of the anomaly equation

$$\langle \vec{p}' s' | Q(0) | \vec{p} s \rangle = \frac{1}{2N_f} \langle \vec{p}' s' | \partial^\mu J_\mu^5(0) | \vec{p} s \rangle = \frac{m_N}{N_f} A(k^2) \bar{u}(\vec{p}' s') i \gamma^5 u(\vec{p} s)$$

$A(k^2) = G_1(k^2) + (k^2/m_N)G_2(k^2)$ can be extracted from a measurement of the matrix element in the left hand side of eq.(19).

In quenched approximation $G_2(k^2)$ has a pole at $k^2 = 0$ [19]. With dynamical quarks this is not the case and, for sufficiently small k^2 $G_1(0)$ or $\Delta\Sigma$ [20] can be determined.

Our preliminary result is $\Delta\Sigma = 0.05 \pm 0.05$. Again the error is underestimated due to the bad sampling of topological charge discussed above.

6 Discussion.

Our conclusions are the following.

A reliable determination of the topological susceptibility χ is possible on the lattice. All existing definitions of lattice topological charge density give results consistent with each other if proper renormalizations are performed. The value of χ for $SU(2)$ is larger than the prediction[2] but the value for $SU(3)$ is consistent with it. The value of $m_{\eta'}$ is well explained by the anomaly. Above the deconfining transition χ drops to zero, more rapidly for $SU(3)$ than for $SU(2)$ [14]. In full QCD preliminary results are in agreement with expectations for χ and χ' and for the spin content of the proton. However a better sampling of topological sectors is needed. In fact the incapability of the hybrid montecarlo to thermalize topology could affect not only the measurement of quantities directly related to it, like the one we considered, but any other measurement on the lattice. Work is in progress to overcome this difficulty.

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